B.Sc Part II (Hons)

Isomorphism Theorem

Theorem. If H and K both are normal subgroups of a group G such that $H \subset K$, then
$$G/K \cong (G/H)/(K/H)$$

Proof: — Let H and K be normal subgroups of a group G such that $H \subset K$ so that the symbols (G/K) and (G/H) are meaningful.

To prove that $G/K \cong (G/H)/(K/H)$

Our assumption implies that K/H is a normal subgroups of G/H by the ~~previous~~ theorem.

Define a map $f : G/H \to G/K$ by the formula
$$f(Hx) = Kx \quad \forall x \in G.$$

Now we shall show that f is a homomorphism onto.

(i) To prove that f is well defined.
$$Hx = Hy,\ x, y \in G \Rightarrow xy^{-1} \in H \subset K$$
$$\Rightarrow xy^{-1} \in K \Rightarrow Kx = Ky$$
$$\Rightarrow f(Hx) = f(Hy)$$
Hence the result (i)

(ii) To prove f is a homomorphism
$$x, y \in G \Rightarrow xy \in G \Rightarrow f(Hxy) = Kxy$$
$$= Kx \cdot Ky$$
[For K is normal $\Rightarrow Kxy \Rightarrow Kx \cdot Ky$]
$$\Rightarrow f(Hxy) = (Kx)(Ky) = f(Hx)\,f(Hy)$$
$\Rightarrow f$ is a homomorphism.

(iii) To prove f is onto.

Given that and, $Kx \in G/K \Rightarrow x \in G$
$\Rightarrow Hx \in G/H$ such that $f(Hx) = Kx$.

Hence the result (iii)

(iv) To prove that kernel of f is K/H.

$$\ker f = \{Hx \in G/H : f(Hx) = K\}$$

K is the identity of G/K

Any $Hx \in \ker f \Rightarrow f(Hx) = K$

$\Rightarrow Kx = K$ by def. of f

$\Rightarrow x \in K \Rightarrow Hx \in K/H$

Any $Hx \in \ker f \Rightarrow Hx \in K/H$.

This proves that $\ker f \subset K/H$ ——— (1)

Again, Any $Hx \in K/H \Rightarrow x \in K \Rightarrow Kx = K \Rightarrow f(Hx) = K$

$\Rightarrow Hx \in \ker f$

Thus $\Rightarrow K/H \subset \ker f$ ——— (2)

Combining (1) and (2) $K/H = \ker f$ ———

Finally we have shown that $f : G/H \xrightarrow{onto} G/K$ is a homomorphism with $\ker f = K/H$.

Also K/H is a normal subgroup of G/H. Hence by the theorem of homomorphism

$$G/K \cong (G/H)/(K/H)$$

Proved.

Normal Subgroup :- A subgroup H of a group G is called a normal subgroup of G iff

$xhx^{-1} \in H \quad \forall x \in G$ and $\forall h \in H$

i.e. iff $xHx^{-1} \subset H \quad \forall x \in G$

The symbol "$H \trianglelefteq G$" is read as H is a normal subgroup of the group G.

Example – If $E = \{2n : n \in Z\}$, then E is clearly subgroup of $(Z, +)$. Again if $x \in Z$, $h \in E$, then $h = 2n$, $n \in Z$ and so

$xhx^{-1} = x + h - x = x + 2n - x = 2n = h \in E$

$\therefore E$ is normal subgroup of $(Z, +)$.

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